

BIDUALITY AND DENSITY IN LIPSCHITZ FUNCTION SPACES

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ABSTRACT. For pointed compact metric spaces (X, d) , we address the biduality problem as to when the space of Lipschitz functions $\text{Lip}_0(X, d)$ is isometrically isomorphic to the bidual of the space of little Lipschitz function $\text{lip}_0(X, d)$, and show that this is the case whenever the closed unit ball of $\text{lip}_0(X, d)$ is dense in the closed unit ball of $\text{Lip}_0(X, d)$ with respect to the topology of pointwise convergence. Then we apply our density criterion to prove in an alternate way the real version of a classical result which asserts that $\text{Lip}_0(X, d^\alpha)$ is isometrically isomorphic to $\text{lip}_0(X, d^\alpha)^{**}$ for any $\alpha \in (0, 1)$.

INTRODUCTION

Let (X, d) be a pointed compact metric space with a base point denoted by 0 and let \mathbb{K} be the field of real or complex numbers. The Lipschitz space $\text{Lip}_0(X, d)$ is the Banach space of all Lipschitz functions $f: X \rightarrow \mathbb{K}$ for which $f(0) = 0$, endowed with the Lipschitz norm

$$\text{Lip}_d(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$

A Lipschitz function $f: X \rightarrow \mathbb{K}$ satisfying the local flatness condition:

$$\lim_{t \rightarrow 0} \sup_{0 < d(x, y) < t} \frac{\|f(x) - f(y)\|}{d(x, y)} = 0,$$

is called a little Lipschitz function, and the little Lipschitz space $\text{lip}_0(X, d)$ is the closed subspace of $\text{Lip}_0(X, d)$ formed by all little Lipschitz functions. Furthermore, $\text{Lip}_0^{\mathbb{R}}(X, d)$ and $\text{lip}_0^{\mathbb{R}}(X, d)$ are the real subspaces of all real-valued functions in $\text{Lip}_0(X, d)$ and $\text{lip}_0(X, d)$, respectively. These spaces have been largely investigated along the time. See the Weaver's book [10] for references and a complete study.

The biduality problem as to when $\text{Lip}_0(X, d)$ is isometrically isomorphic to $\text{lip}_0(X, d)^{**}$ has an interesting history (see [10, p. 99, Notes 3.3] and also [8, 6. Duality]). In this note, we address this question in a similar way as Bierstedt and Summers [2] do for studying the biduals of weighted Banach spaces of analytic functions, and we prove that $\text{Lip}_0(X, d)$ is isometrically isomorphic to $\text{lip}_0(X, d)^{**}$ if and only if the closed unit ball of $\text{lip}_0(X, d)$ is dense in the closed unit ball of $\text{Lip}_0(X, d)$ with respect to the topology of pointwise convergence τ_p . This density condition is equivalent to requiring that for each $f \in \text{Lip}_0(X, d)$ with $\text{Lip}_d(f) \leq 1$, there exists a sequence $\{f_n\}$ in $\text{lip}_0(X, d)$ with $\text{Lip}_d(f_n) \leq 1$ for all $n \in \mathbb{N}$ such that $\{f_n(x)\}$ converges to $f(x)$ as $n \rightarrow \infty$ for every $x \in X$. Then we apply our density criterion to prove in an alternate way the real version of a classical Johnson's result [7] (see also [1, 5, 10]) which asserts that $\text{Lip}_0(X, d^\alpha)$ is isometrically isomorphic to $\text{lip}_0(X, d^\alpha)^{**}$ for any $\alpha \in (0, 1)$.

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THE RESULTS

Johnson [7] proved that the closed linear subspace of $\text{Lip}_0(X, d)^*$ spanned by the evaluation functionals $\delta_x: \text{Lip}_0(X, d) \rightarrow \mathbb{K}$, given by $\delta_x(f) = f(x)$ with $x \in X$, is a predual of $\text{Lip}_0(X, d)$. The terminology Lipschitz-free Banach space of X and the notation $\mathcal{F}(X)$ for this predual of $\text{Lip}_0(X, d)$ are due to Godefroy and Kalton [6]. Namely, the evaluation map $Q_X: \text{Lip}_0(X, d) \rightarrow \mathcal{F}(X)^*$ defined by

$$Q_X(f)(\gamma) = \gamma(f) \quad (f \in \text{Lip}_0(X, d), \gamma \in \mathcal{F}(X))$$

is the natural isometric isomorphism. As usual, B_E will denote the closed unit ball of a Banach space E .

Theorem 1. *Let (X, d) be a pointed compact metric space.*

- (i) *The restriction map $R_X: \mathcal{F}(X) \rightarrow \text{lip}_0(X, d)^*$ defined by*

$$R_X(\gamma)(f) = \gamma(f) \quad (f \in \text{lip}_0(X, d), \gamma \in \mathcal{F}(X)),$$

is a nonexpansive linear surjective map.

- (ii) *R_X is an isometric isomorphism from $\mathcal{F}(X)$ onto $\text{lip}_0(X, d)^*$ if and only if $B_{\text{lip}_0(X, d)}$ is dense in $B_{\text{Lip}_0(X, d)}$ with respect to the topology of pointwise convergence.*

Proof. (i) Since $\mathcal{F}(X) \subset \text{Lip}_0(X, d)^*$, it is clear that R_X is a linear map from $\mathcal{F}(X)$ into $\text{lip}_0(X, d)^*$ satisfying $\|R_X(\gamma)\| \leq \|\gamma\|$ for all $\gamma \in \mathcal{F}(X)$. We next prove that R_X is surjective. To this end, let us recall that De Leeuw's map $\Phi: \text{Lip}_0(X, d) \rightarrow C_b(\tilde{X})$ given by

$$\Phi(f)(x, y) = \frac{f(x) - f(y)}{d(x, y)} \quad (f \in \text{Lip}_0(X, d), (x, y) \in \tilde{X}),$$

where $\tilde{X} = \{(x, y) \in X^2: x \neq y\}$, is a linear isometry of $\text{Lip}_0(X, d)$ into $C_b(\tilde{X})$, the Banach space of bounded continuous scalar-valued functions on \tilde{X} with the supremum norm, and the image of $\text{lip}_0(X, d)$ is contained in $C_0(\tilde{X})$, the closed subspace of functions which vanish at infinity. See, for example, [10, Theorem 2.1.3 and Proposition 3.1.2].

Take $\gamma \in \text{lip}_0(X, d)^*$. The functional $T: \Phi(\text{lip}_0(X, d)) \rightarrow \mathbb{K}$, defined by $T(\Phi(f)) = \gamma(f)$ for all $f \in \text{lip}_0(X, d)$, is linear, continuous and $\|T\| = \|\gamma\|$. By the Hahn–Banach theorem, there exists a continuous linear functional $\tilde{T}: C_0(\tilde{X}) \rightarrow \mathbb{K}$ such that $\tilde{T}(\Phi(f)) = T(\Phi(f))$ for all $f \in \text{lip}_0(X, d)$ and $\|\tilde{T}\| = \|T\|$. Now, by the Riesz representation theorem, there exists a finite and regular Borel measure μ on \tilde{X} with total variation $\|\mu\| = \|\tilde{T}\|$ such that

$$\tilde{T}(g) = \int_{\tilde{X}} g d\mu \quad (g \in C_0(\tilde{X})),$$

and thus

$$\gamma(f) = \int_{\tilde{X}} \Phi(f) d\mu \quad (f \in \text{lip}_0(X, d)).$$

If we now define

$$\tilde{\gamma}(f) = \int_{\tilde{X}} \Phi(f) d\mu \quad (f \in \text{Lip}_0(X, d)),$$

it is clear that $\tilde{\gamma} \in \text{Lip}_0(X, d)^*$ and $\tilde{\gamma}(f) = \gamma(f)$ for all $f \in \text{lip}_0(X, d)$. It remains to show that $\tilde{\gamma}$ is τ_p -continuous on $B_{\text{Lip}_0(X, d)}$. Thus, let $\{f_i\}$ be a net in $B_{\text{Lip}_0(X, d)}$ which converges pointwise on X to zero. Then $\{\Phi(f_i)\}$ converges pointwise on \tilde{X} to zero and, since $|\Phi(f_i)(x, y)| \leq \|\Phi(f_i)\|_\infty = \text{Lip}_d(f_i) \leq 1$ for all $i \in I$ and for all $(x, y) \in \tilde{X}$, it follows that $\{\tilde{\gamma}(f_i)\}$ converges to 0 by the Lebesgue's bounded convergence theorem. This completes the proof of (i).

(ii) Assume that $B_{\text{lip}_0(X,d)}$ is τ_p -dense in $B_{\text{Lip}_0(X,d)}$. Fix $\gamma \in \mathcal{F}(X)$ and let $f \in B_{\text{Lip}_0(X,d)}$. Then there exists a net $\{f_i\}$ in $B_{\text{lip}_0(X,d)}$ which converges to f in the topology of pointwise convergence. Since γ is τ_p -continuous on $B_{\text{Lip}_0(X,d)}$ and it is satisfied that

$$|\gamma(f_i)| = |R_X(\gamma)(f_i)| \leq \|R_X(\gamma)\| \text{Lip}_d(f_i) \leq \|R_X(\gamma)\|$$

for all $i \in I$, it follows that $|\gamma(f)| \leq \|R_X(\gamma)\|$ and so $\|\gamma\| \leq \|R_X(\gamma)\|$. Now, taking into account (i) we conclude that R_X is an isometric isomorphism from $\mathcal{F}(X)$ onto $\text{lip}_0(X,d)^*$.

Conversely, if $B_{\text{lip}_0(X,d)}$ is not τ_p -dense in $B_{\text{Lip}_0(X,d)}$, by the Hahn–Banach theorem there exist a function $g \in B_{\text{Lip}_0(X,d)}$ and a τ_p -continuous linear functional γ on $\text{Lip}_0(X,d)$ such that $|\gamma(f)| \leq 1$ for all $f \in B_{\text{lip}_0(X,d)}$ and $|\gamma(g)| > 1$. Since $\gamma \in \mathcal{F}(X)$ and $\|R_X(\gamma)\| = \|\gamma|_{\text{lip}_0(X,d)}\| \leq 1 < |\gamma(g)| \leq \|\gamma\|$, then R_X is not an isometry. \square

We are now ready to obtain the main result of this note.

Theorem 2. *Let (X,d) be a pointed compact metric space. Then the following are equivalent:*

- (i) $\text{Lip}_0(X,d)$ is isometrically isomorphic to $\text{lip}_0(X,d)^{**}$.
- (ii) $B_{\text{lip}_0(X,d)}$ is dense in $B_{\text{Lip}_0(X,d)}$ with respect to the weak* topology.
- (iii) $B_{\text{lip}_0(X,d)}$ is dense in $B_{\text{Lip}_0(X,d)}$ with respect to the topology of pointwise convergence.
- (iv) For each $f \in B_{\text{Lip}_0(X,d)}$, there exists a sequence $\{f_n\}$ in $B_{\text{lip}_0(X,d)}$ such that $\{f_n(x)\}$ converges to $f(x)$ as $n \rightarrow \infty$ for every $x \in X$.

Proof. If (i) holds, then (ii) follows by the Goldstine theorem; but (ii) is the same as (iii) since the weak* topology agrees with the topology of pointwise convergence on bounded subsets of $\text{Lip}_0(X,d)$ by [7, Corollary 4.4]. If (iii) is true, then R_X^* is an isometric isomorphism from $\text{lip}_0(X,d)^{**}$ onto $\mathcal{F}(X)^*$ by Theorem 1, hence the composition $Q_X^{-1} \circ R_X^*$ is an isometric isomorphism from $\text{lip}_0(X,d)^{**}$ onto $\text{Lip}_0(X,d)$ and so we obtain (i).

In order to prove that (ii) is equivalent to (iv), notice that, by [7, Corollary 4.4], the family of sets

$$U(f_0; n, x_1, \dots, x_n, \varepsilon) := \{f \in B_{\text{Lip}_0(X,d)} : |f(x_i) - f_0(x_i)| < \varepsilon, \forall i = 1, \dots, n\}$$

with $f_0 \in B_{\text{Lip}_0(X,d)}$, $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\varepsilon > 0$, is a basis of the relative weak* topology on $B_{\text{Lip}_0(X,d)}$.

Suppose now that (ii) holds and let $f_0 \in B_{\text{Lip}_0(X,d)}$. Given $x \in X$ and $n \in \mathbb{N}$, the set $U(f_0; 1, x, 1/n)$ is a weak* neighborhood of f_0 relative to $B_{\text{Lip}_0(X,d)}$. Then, by (ii), for each $n \in \mathbb{N}$ there exists $f_n \in B_{\text{lip}_0(X,d)}$ such that $f_n \in U(f_0; 1, x, 1/n)$, that is, $|f_n(x) - f_0(x)| < 1/n$. Hence $\{f_n(x)\}$ converges to $f_0(x)$ as $n \rightarrow \infty$ and we conclude that (ii) implies (iv). Conversely, assume that (iv) is valid and let $f_0 \in B_{\text{Lip}_0(X,d)}$. Take $U(f_0; p, x_1, \dots, x_p, \varepsilon)$ with $p \in \mathbb{N}$, $x_1, \dots, x_p \in X$ and $\varepsilon > 0$. By (iv), there is a sequence $\{f_n\}$ in $B_{\text{lip}_0(X,d)}$ such that $\{f_n(x)\}$ converges to $f_0(x)$ as $n \rightarrow \infty$ for every $x \in X$. In particular, for each $i \in \{1, \dots, p\}$, there is a $m_i \in \mathbb{N}$ for which $|f_n(x_i) - f_0(x_i)| < \varepsilon$ whenever $n \geq m_i$. Now, if $m = \max\{m_1, \dots, m_p\}$, then $f_m \in U(f_0; p, x_1, \dots, x_p, \varepsilon)$ and (ii) follows. \square

It is known that $\text{Lip}_0(X,d)$ is isometrically isomorphic to $\text{lip}_0(X,d)^{**}$ for a large class of metric spaces (X,d) as, for example, the Hölder spaces (X, d^α) , $0 < \alpha < 1$ [1, 5, 7].

Remark 3. *The proof of Theorem 2 shows that if one of its statements holds, then the map $Q_X^{-1} \circ R_X^*$ is an isometric isomorphism from $\text{lip}_0(X,d)^{**}$ onto $\text{Lip}_0(X,d)$. For any $\phi \in \text{lip}_0(X,d)^{**}$ and $x \in X$, an*

easy verifications yields

$$\begin{aligned}
(Q_X^{-1} \circ R_X^*)(\phi)(x) &= \delta_x((Q_X^{-1} \circ R_X^*)(\phi)) \\
&= Q_X((Q_X^{-1} \circ R_X^*)(\phi))(\delta_x) \\
&= Q_X(Q_X^{-1}(R_X^*(\phi)))(\delta_x) \\
&= R_X^*(\phi)(\delta_x) \\
&= \phi(R_X(\delta_x)) \\
&= \phi(\delta_x).
\end{aligned}$$

This identification is the same as that obtained by De Leeuw [5], Johnson [7] and Bade, Curtis and Dales [1] between the spaces $\text{Lip}_0(X, d^\alpha)$ and $\text{lip}_0(X, d^\alpha)^{**}$ ($0 < \alpha < 1$).

The pointwise approximation condition given by the assertion (iv) of Theorem 2 can be verified to recover two classical results about the biduality problem of $\text{Lip}_0(X, d^\alpha)$ ($0 < \alpha < 1$). The former is due to Ciesielski [4] and the latter to De Leeuw [5].

Example 4. Let $\alpha \in (0, 1)$ and let $[0, 1]$ be the unit interval with the usual metric d . Then $\text{Lip}_0([0, 1], d^\alpha)$ is isometrically isomorphic to $\text{lip}_0([0, 1], d^\alpha)^{**}$.

Proof. Fix $f \in B_{\text{Lip}_0([0, 1], d^\alpha)}$ and, for each $n \in \mathbb{N}$, let $B_n(f, \cdot)$ denote the n th Bernstein polynomial for f defined by

$$B_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad (x \in [0, 1]).$$

Then $B_n(f, \cdot)$ also belongs to $B_{\text{Lip}_0([0, 1], d^\alpha)}$ (see [3] for an elementary proof) while the fact that

$$\begin{aligned}
|B_n(f, x) - B_n(f, y)| &\leq \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) \right| \binom{n}{k} |x^k (1-x)^{n-k} - y^k (1-y)^{n-k}| \\
&\leq |x - y| \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) \right| \binom{n}{k} 2n
\end{aligned}$$

for all $x, y \in [0, 1]$ shows that $B_n(f, \cdot) \in B_{\text{lip}_0([0, 1], d^\alpha)}$. Since $\{B_n(f, \cdot)\}_{n \in \mathbb{N}}$ converge to f uniformly on $[0, 1]$, the example is proved by Theorem 2. \square

Example 5. Let $0 < \alpha < 1$ and let \mathbb{T} be the quotient additive group $\mathbb{R}/2\pi\mathbb{Z}$ with the distance

$$d(t + 2\pi\mathbb{Z}, s + 2\pi\mathbb{Z}) = \min \{|t - s|, |t - s - 2\pi|, |t - s + 2\pi|\} \quad (t, s \in [0, 2\pi)).$$

Then $\text{Lip}_0(\mathbb{T}, d^\alpha)$ is isometrically isomorphic to $\text{lip}_0(\mathbb{T}, d^\alpha)^{**}$.

Proof. We apply similar arguments to those of [5, Lemma 2.8] and use some results from harmonic analysis (see [9]). We identify each equivalence class $t + 2\pi\mathbb{Z}$ with the point $t \in [0, 2\pi)$. Let $f \in B_{\text{Lip}_0(\mathbb{T}, d^\alpha)}$. For each $n \in \mathbb{N}$, let K_n be the Fejér kernel defined by

$$K_n(t) = \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \frac{1}{n+1} \left(\frac{\sin \frac{n+1}{2} t}{\sin \frac{t}{2}}\right)^2 \quad (t \in [0, 2\pi)).$$

Then the convolution

$$(K_n * f)(t) = \frac{1}{2\pi} \int_0^{2\pi} K_n(\tau) f(t - \tau) d\tau \quad (t \in [0, 2\pi))$$

coincides with the Fejér mean

$$\sigma_n(f, t) = \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) \widehat{f}(j) e^{ijt} \quad (t \in [0, 2\pi)),$$

where $\widehat{f}(j)$ is the j th Fourier coefficient of f . Given $t, s \in [0, 2\pi)$, we have

$$\begin{aligned} |\sigma_n(f, t) - \sigma_n(f, s)| &\leq \sum_{j=-n}^n \left|1 - \frac{|j|}{n+1}\right| |\widehat{f}(j)| |e^{ijt} - e^{ijs}| \\ &\leq \sum_{j=-n}^n \left|1 - \frac{|j|}{n+1}\right| \frac{\pi^\alpha |j|^{1-\alpha}}{2} (4\pi n)^n (e-1) d(t, s) \end{aligned}$$

and therefore $\sigma_n(f, \cdot) \in \text{lip}_0(\mathbb{T}, d^\alpha)$. Moreover,

$$\begin{aligned} |\sigma_n(f, t) - \sigma_n(f, s)| &= |(K_n * f)(t) - (K_n * f)(s)| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |K_n(\tau)| |f(t - \tau) - f(s - \tau)| d\tau \\ &\leq \text{Lip}_{d^\alpha}(f) d(t, s)^\alpha \frac{1}{2\pi} \int_0^{2\pi} K_n(\tau) d\tau \\ &= \text{Lip}_{d^\alpha}(f) d(t, s)^\alpha, \end{aligned}$$

and so $\text{Lip}_{d^\alpha}(\sigma_n(f, \cdot)) \leq \text{Lip}_{d^\alpha}(f) \leq 1$. Now take $\beta_n(f, \cdot) = \sigma_n(f, \cdot) - \sigma_n(f, 0)$ which is in $B_{\text{lip}_0(\mathbb{T}, d^\alpha)}$. By the Fejér theorem, $\{\sigma_n(f, \cdot)\}_{n \in \mathbb{N}}$ converges pointwise on \mathbb{T} to f , and so does $\{\beta_n(f, \cdot)\}_{n \in \mathbb{N}}$. Then the desired conclusion follows from Theorem 2. \square

Our density criterion serves to give another proof of the real version of an important result by Johnson [7, Theorem 4.7] and Bade, Curtis and Dales [1, Theorem 3.5].

Corollary 6. *Let (X, d) be a pointed compact metric space and let $\alpha \in (0, 1)$. Then $\text{Lip}_0^\mathbb{R}(X, d^\alpha)$ is isometrically isomorphic to $\text{lip}_0^\mathbb{R}(X, d^\alpha)^{**}$.*

Proof. Let $f \in B_{\text{Lip}_0^\mathbb{R}(X, d^\alpha)}$. We claim that for each $n \in \mathbb{N}$ and each finite set $F \subset X$, there exists a function $h \in \text{lip}^\mathbb{R}(X, d^\alpha)$ such that $\text{Lip}_{d^\alpha}(h) \leq 1 + 1/n$ and $h(x) = f(x)$ for all $x \in F$. The notation $\text{lip}^\mathbb{R}(X, d^\alpha)$ and later $\text{Lip}^\mathbb{R}(X, d^\gamma)$ might be self-explanatory.

Consider $F = \{x_1, \dots, x_m\}$ for some $m \in \mathbb{N}$ and there is no loss of generality in assuming that $f(x_m) \leq f(x_{m-1}) \leq \dots \leq f(x_1)$. If $m = 1$, we set $h(x) = f(x_1)$ for all $x \in X$. Now let $m \geq 2$ and we also may assume $f \geq 0$, for otherwise we can replace f by $f + \|f\|_\infty$. Let

$$\begin{aligned} \gamma &= \min \left(\left\{ \alpha + \frac{e \ln(1 + \frac{1}{n})}{\text{diam}(X)} d(x_j, x_k) : j, k \in \{1, \dots, m\}, j \neq k \right\} \cup \{1\} \right), \\ \rho &= \max \left\{ \frac{|f(x_k) - f(x_j)|}{d(x_k, x_j)^\gamma} : j, k \in \{1, \dots, m\}, j \neq k \right\}. \end{aligned}$$

For each $j \in \{1, \dots, m\}$, define $g_j : X \rightarrow \mathbb{R}$ by

$$g_j(x) = \max \{f(x_j) - \rho d(x_j, x)^\gamma, 0\}.$$

Notice that $0 < \alpha < \gamma \leq 1$ and therefore $g_j \in \text{Lip}^\mathbb{R}(X, d^\gamma) \subset \text{lip}^\mathbb{R}(X, d^\alpha)$ with

$$\text{Lip}_{d^\alpha}(g_j) \leq \text{Lip}_\gamma(g_j) \text{diam}(X)^{\gamma-\alpha} \leq \rho \text{diam}(X)^{\gamma-\alpha}.$$

We now check that the function $h = \max\{g_1, \dots, g_m\}$ satisfies the required conditions. It is known that h is in $\text{lip}^{\mathbb{R}}(X, d^\alpha)$ and $\text{Lip}_{d^\alpha}(h) \leq \max\{\text{Lip}_{d^\alpha}(g_1), \dots, \text{Lip}_{d^\alpha}(g_m)\}$. Now, given $j \in \{1, \dots, m\}$, for some $k, i \in \{1, \dots, m\}$ with $k \neq i$, we have

$$\begin{aligned} \text{Lip}_{d^\alpha}(g_j) &\leq \rho \text{diam}(X)^{\gamma-\alpha} \\ &= \frac{|f(x_k) - f(x_i)|}{d(x_k, x_i)^\gamma} \text{diam}(X)^{\gamma-\alpha} \\ &\leq \text{Lip}_{d^\alpha}(f) \left(\frac{\text{diam}(X)}{d(x_k, x_i)} \right)^{\gamma-\alpha} \\ &\leq \left(\frac{\text{diam}(X)}{d(x_k, x_i)} \right)^{\frac{e \ln(1+\frac{1}{n})}{\text{diam}(X)} d(x_k, x_i)} \\ &\leq 1 + \frac{1}{n}. \end{aligned}$$

The last inequality follows from the fact that the function $t \mapsto (t/\text{diam}(X))^{te \ln(1+1/n)/\text{diam}(X)}$ for all $t > 0$ has a minimum value of $1/(1+1/n)$. Hence $\text{Lip}_{d^\alpha}(h) \leq 1 + 1/n$ as required. Now let $j, k \in \{1, \dots, m\}$. If $j \leq k$, it is immediate that $g_k(x_j) \leq f(x_k) \leq f(x_j) = g_j(x_j)$, whereas that if $k < j$, we have $|f(x_k) - f(x_j)|/d(x_k, x_j)^\gamma \leq \rho$, hence $f(x_k) - \rho d(x_k, x_j)^\gamma \leq f(x_j)$ and thus $g_k(x_j) \leq g_j(x_j)$. Therefore $h(x_j) = g_j(x_j) = f(x_j)$ for all $j \in \{1, \dots, m\}$. The claim follows.

Now fix $n \in \mathbb{N}$ and, for each $x \in X$, let $B(x, 1/n) = \{y \in X : d(y, x)^\alpha < 1/n\}$. By the compactness of X , there is a finite subset F_n of X such that $X = \bigcup_{x \in F_n} B(x, 1/n)$. We can suppose that the base point $0 \in X$ is in F_n , for otherwise take the finite set $F_n \cup \{0\}$. By the claim, there exists a function $h_n \in \text{lip}^{\mathbb{R}}(X, d^\alpha)$ such that $\text{Lip}_{d^\alpha}(h_n) \leq 1 + 1/n$ and $h_n(x) = f(x)$ for all $x \in F_n$. Hence $h_n \in \text{lip}_0^{\mathbb{R}}(X, d^\alpha)$. To prove that the sequence $\{h_n\}$ converges pointwise on X to f , let $x \in X$. For each $n \in \mathbb{N}$, choose $y_n \in F_n$ such that $d(x, y_n)^\alpha < 1/n$. Note that $h_n(y_n) = f(y_n)$ and thus

$$\begin{aligned} |f(x) - h_n(x)| &\leq |f(x) - f(y_n)| + |f(y_n) - h_n(x)| \\ &\leq |f(x) - f(y_n)| + |h_n(y_n) - h_n(x)| \\ &\leq (\text{Lip}_{d^\alpha}(f) + \text{Lip}_{d^\alpha}(h_n)) d(x, y_n)^\alpha \\ &\leq \left(2 + \frac{1}{n} \right) \frac{1}{n}. \end{aligned}$$

Hence the sequence $\{h_n(x)\}$ converges to $f(x)$ as $n \rightarrow \infty$. Finally, let $r_n = \max\{1, \text{Lip}_{d^\alpha}(h_n)\}$ and $f_n = h_n/r_n$ for each $n \in \mathbb{N}$. It is clear that $\{f_n\}$ is a sequence in $B_{\text{lip}_0^{\mathbb{R}}(X, d^\alpha)}$ that converges pointwise to f on X . Then the corollary follows from Theorem 2. \square

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